REPORT ON LOCALLY FINITE TRIANGULATED CATEGORIES

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ABSTRACT. The basic properties of locally finite triangulated categories are discussed. The focus is on Auslander–Reiten theory and the lattice of thick subcategories.

CONTENTS

1. INTRODUCTION

This is a report on a particular class of triangulated categories. A triangulated category T is said to be *locally finite* if every cohomological functor from T or its opposite category T^{op} into the category of abelian groups is a direct sum of representable functors.

We present a number of basic results for such triangulated categories. Some of these results seem to be new, but we include also results which are variations or generalisations of known results. Thus our aim is to provide the foundations for studying the locally finite triangulated categories.

A basic tool for understanding a triangulated category T is the collection of representable functors $\mathrm{Hom}_{\mathsf{T}}(-, X) \colon \mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}$ where X runs through the objects of T. We show that T is locally finite if and only if each representable functor is of finite length. This property justifies the term 'locally finite' which is due to Xiao and Zhu [51] in the triangulated context and goes back to Gabriel [21].

An important thread in the study of locally finite triangulated categories is the use of Auslander–Reiten theory. The principal idea is to analyse for each object X in T the radical filtration

$$\ldots \subseteq \mathrm{Rad}_{\mathsf{T}}^2(-, X) \subseteq \mathrm{Rad}_{\mathsf{T}}^1(-, X) \subseteq \mathrm{Rad}_{\mathsf{T}}^0(-, X) = \mathrm{Hom}_{\mathsf{T}}(-, X)$$

which is finite when T is locally finite. Some of this information is encoded in the Auslander–Reiten quiver of T which can be described fairly explicitly.

Another intriguing invariant of a triangulated category is the lattice of thick subcategories. Assuming locally finiteness, we show that the inclusion of each thick subcategory admits a left and a right adjoint. In fact, the lattice has interesting symmetries and is even finite if the category is finitely generated.

The results presented here are most complete when the category is *simply connected*, that is, the Auslander–Reiten quiver is connected and contains no oriented cycle. For instance, the lattice of thick subcategories is in this case isomorphic to the lattice of non-crossing partitions associated to some diagram of Dynkin type.

Using covering theory, the study of a general locally finite triangulated category can often be reduced to the simply connected case. For this direction we refer to recent work of Amiot [1] and Köhler [36].

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2. Locally noetherian triangulated categories

In this section, locally noetherian and locally finite triangulated categories are introduced. We provide various characterisations and a host of examples. Then we establish the existence of adjoints for inclusions of thick subcategories.

Throughout this work let T denote a triangulated category with suspension Σ.

The abelianisation of a triangulated category. Following Freyd [20, §3] and Verdier [48, II.3], we consider the *abelianisation* $\mathbf{A}(\mathsf{T})$ of a triangulated category T which is by definition the abelian category consisting of all additive functors $F\colon \mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}$ into the category of abelian groups that fit into an exact sequence

$$\mathrm{Hom}_{\mathsf{T}}(-, X) \longrightarrow \mathrm{Hom}_{\mathsf{T}}(-, Y) \longrightarrow F \longrightarrow 0.$$

The fully faithful Yoneda functor $H\colon \mathsf{T} \to \mathbf{A}(\mathsf{T})$ taking an object X to the representable functor $\mathrm{Hom}_{\mathsf{T}}(-, X)$ is the universal cohomological functor starting in T, that is, each cohomological functor $\mathsf{T} \to \mathsf{A}$ to an abelian category A factors essentially uniquely through H. Observe that the functor taking $\mathrm{Hom}_{\mathsf{T}}(-, X)$ to $\mathrm{Hom}_{\mathsf{T}}(X, -)$ induces an equivalence

$$(2.1) \qquad\qquad\qquad \mathbf{A}(\mathsf{T})^{\mathrm{op}} \overset{\sim}{\longrightarrow} \mathbf{A}(\mathsf{T}^{\mathrm{op}}).$$

This is an immediate consequence of the universal property of the Yoneda functor.

Locally noetherian triangulated categories. Given an essentially small triangulated category T, we use its abelianisation to formulate a useful finiteness condition; see also [9]. We say that T is *locally noetherian*[1] if the equivalent conditions of the following theorem are satisfied.

Theorem 2.1. *For an essentially small triangulated category T the following conditions are equivalent.*

(1) *Every cohomological functor $\mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}$ into the category of abelian groups is a direct sum of representable functors.*

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[1]The terminology refers to the equivalent fact that the abelian category of additive functors $\mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}$ is locally noetherian in the sense of [21, II.4].

(2) *Every object in* T *is a finite coproduct of indecomposable objects with local endomorphism rings, and for every sequence* $X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} X_3 \xrightarrow{\phi_3} \ldots$ *of non-isomorphisms between indecomposable objects there exists some number* n *such that* $\phi_n \ldots \phi_2 \phi_1 = 0$.

(3) *Idempotents in* T *split and every object of the abelianization* $\mathbf{A}(\mathsf{T})$ *is noetherian, that is, every ascending chain of subobjects in* $\mathbf{A}(\mathsf{T})$ *eventually stabilises.*

Proof. We view the additive category T as a *ring with several objects* and think of additive functors $\mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}$ as T-modules. Note that a T-module is flat if and only if it is a cohomological functor; see [37, Lemma 2.7]. Bass has characterised the rings for which every flat module is projective. This can be generalised to modules over rings with several objects, see [30, Theorem B.12], and yields the equivalence of conditions (1) and (2).

Recall that a module M is *fp-injective* if $\mathrm{Ext}^1(-, M)$ vanishes on all finitely presented modules. Note that T as a ring with several objects is *noetherian* (that is, each representable functor $\mathrm{Hom}_{\mathsf{T}}(-, X)$ satisfies the ascending chain condition on subfunctors) if and only if every fp-injective T-module is injective; see [30, Theorem B.17]. The fp-injective T-modules are precisely the cohomological functors $\mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}$, by [37, Lemma 2.7].

Suppose that (1) holds and fix an fp-injective T-module M. Choose an injective envelope $\phi \colon M \to Q$. The snake lemma shows that the cokernel $\mathrm{Coker}\,\phi$ is cohomological. Thus $\mathrm{Coker}\,\phi$ is a direct sum of representable functors and therefore projective; in particular ϕ splits. It follows that M is injective, and therefore T is noetherian. It remains to show that T is idempotent complete. But this is clear because a direct summand of a representable functor is cohomological and therefore representable. Thus (3) holds.

Now suppose that (3) holds. Thus the ascending chain condition holds for chains of finitely presented submodules of modules of the form $\mathrm{Hom}_{\mathsf{T}}(-, X)$. This implies the ascending chain conditions for arbitrary submodules, since each submodule is a union of finitely generated submodules and each finitely generated submodule of a finitely presented one is again finitely presented. It follows that T is noetherian.

Fix a flat T-module M and choose an epimorphism $\phi \colon P \to M$ such that P is projective. The snake lemma shows that the kernel $\mathrm{Ker}\,\phi$ is cohomological. Thus $\mathrm{Ker}\,\phi$ is fp-injective and therefore injective; in particular ϕ splits. It follows that M is projective, and therefore a direct sum of finitely generated projective modules; see [30, Corollary B.13]. The finitely generated projective modules are precisely the representable functors since T has split idempotents. Thus (1) holds. $\qquad\square$

Example 2.2. Fix a field k and denote by A the category of k-linear representations of the quiver

$$\Gamma \colon \quad 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow \cdots$$

that are finite dimensional and have finite support. Then the bounded derived category $\mathbf{D}^b(\mathsf{A})$ is locally noetherian but its opposite category is not.

Indeed, each object in $\mathbf{D}^b(\mathsf{A})$ decomposes into a finite coproduct of indecomposable objects with local endomorphism rings. The indecomposable objects are isomorphic to complexes concentrated in a single degree (thus of the form $X[i]$ with $X \in \mathsf{A}$ and $i \in \mathbb{Z}$) since $\mathrm{Ext}_{\mathsf{A}}^p(-, -) = 0$ for $p > 1$.

Given a sequence $X_1, \ldots, X_r$ of objects in A and a sequence of morphism $X_1[i_1] \to X_2[i_2] \to \cdots \to X_r[i_r]$ in $\mathbf{D}^b(\mathsf{A})$ such that their composite is non-zero, we have

$i_1 \leq \ldots \leq i_r \leq i_1 + 1$. Thus it remains to observe that infinite chains of non-isomorphisms between indecomposable objects in A exist only in one direction. More precisely, for each $n \geq 1$ let A_n denote the full subcategory of representations with support contained in $\{1, \ldots, n\}$. Then any non-zero morphism $X \to Y$ between indecomposable representations has the property that $X \in \mathsf{A}_n$ implies $Y \in \mathsf{A}_n$. This is an immediate consequence of the fact that each indecomposable representation of Γ is, up to isomorphism, of the following form:

$$0 \to \cdots \to 0 \to k \xrightarrow{1} \cdots \xrightarrow{1} k \to 0 \to \cdots$$

On the other hand, there is an obvious chain of proper epimorphisms $\cdots \to X_3 \to X_2 \to X_1$ in A, where X_n denotes the unique indecomposable representation with support $\{1, \ldots, n\}$.

Locally finite triangulated categories. An essentially small triangulated category T is said to be *locally finite* if T and T^{op} are locally noetherian. The first condition means that each representable functor $\mathrm{Hom}_{\mathsf{T}}(-, X)$ is a noetherian T-module. In particular, each subobject belongs to $\mathbf{A}(\mathsf{T})$. Combining the second condition with the equivalence $\mathbf{A}(\mathsf{T})^{\mathrm{op}} \xrightarrow{\sim} \mathbf{A}(\mathsf{T}^{\mathrm{op}})$, it follows that $\mathrm{Hom}_{\mathsf{T}}(-, X)$ is an artinian T-module. Thus locally finite means that each representable functor $\mathrm{Hom}_{\mathsf{T}}(-, X)$ is of finite length as a T-module. In particular, T is locally finite if and only if the category of T-modules is locally finite in the sense of [21, II.4].

Suppose that T is locally finite and fix an object Y. Then there are only finitely many isomorphism classes of indecomposable objects X satisfying $\mathrm{Hom}_{\mathsf{T}}(X, Y) \neq 0$, because $\mathrm{Hom}_{\mathsf{T}}(X, Y) \neq 0$ implies that $\mathrm{Hom}_{\mathsf{T}}(-, X)$ is the projective cover of a composition factor of $\mathrm{Hom}_{\mathsf{T}}(-, Y)$. This observation gives rise to the following characterisation which can be deduced from [3, Theorem 2.12].

Proposition 2.3 (Auslander). *An essentially small triangulated category T with split idempotents is locally finite if and only if for each object Y the following holds:*

    (1) *The object Y decomposes into a finite direct sum of indecomposable objects.*
    (2) *There are only finitely many isomorphism classes of indecomposable objects X satisfying $\mathrm{Hom}_{\mathsf{T}}(X, Y) \neq 0$.*
    (3) *For each indecomposable object X, the $\mathrm{End}_{\mathsf{T}}(X)$-module $\mathrm{Hom}_{\mathsf{T}}(X, Y)$ is of finite length.* $\square$

Examples. We list some examples of triangulated categories that are locally finite. Throughout we fix a field k.

(1) Let T be an essentially small k-linear triangulated category. Suppose that idempotents split and that morphism spaces are finite dimensional. Then T is locally finite if and only if for each object Y there are only finitely many isomorphism classes of indecomposable objects X satisfying $\mathrm{Hom}_{\mathsf{T}}(X, Y) \neq 0$. This follows from Proposition 2.3 and serves as a definition in [51].

(2) Let A be a finite dimensional k-algebra and suppose that k is algebraically closed. Then the bounded derived category $\mathbf{D}^b(\mathrm{mod}\, A)$ of the category of finite dimensional A-modules is locally finite if and only if it is triangle equivalent to $\mathbf{D}^b(\mathrm{mod}\, k\Gamma)$ for some path algebra $k\Gamma$ of a finite quiver Γ such that its underlying diagram is a disjoint union of diagrams of Dynkin type; see [25, §5] and [10, Theorem 12.20].

(3) Let A be an essentially small hereditary abelian category. Then the derived category $\mathbf{D}^b(\mathsf{A})$ is locally finite if and only if A satisfies the conditions in Proposition 2.3. This follows from the fact that each indecomposable object is isomorphic to a complex that is concentrated in a single degree. If A is the category of finitely

generated modules over an artinian ring, then this condition means that the ring is of finite representation type.

(4) Let A be a noetherian ring and suppose that A is *Gorenstein*, that is, A has finite injective dimension as an A-module. Denote by $\mathrm{MCM}(A)$ the category of finitely generated A-modules X that are *maximal Cohen–Macaulay*, which means that $\mathrm{Ext}_A^i(X, A) = 0$ for all $i > 0$. This is an exact Frobenius category, and the stable category $\underline{\mathrm{MCM}}(A)$ modulo all morphisms that factor through a projective object is a triangulated category [16].

If A is a finite dimensional and self-injective k-algebra, then all A-modules are maximal Cohen–Macaulay, and $\underline{\mathrm{MCM}}(A)$ is locally finite if and only if A is of *finite representation type*, that is, there are only finitely many isomorphism classes of indecomposable A-modules. These algebras have been classified [44, 49].

If A is a commutative complete local ring, then A is by definition of *finite Cohen–Macaulay type* if there there are only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay modules over A. In that case $\underline{\mathrm{MCM}}(A)$ is locally finite. There is a whole theory describing such rings and a parallel theory for graded Gorenstein algebras; see [50].

(5) Let Γ be a quiver of Dynkin type. Then the orbit category $\mathbf{D}^b(\mathrm{mod}\, k\Gamma)/G$ of the derived category with respect to an appropriate group G of automorphisms is a locally finite triangulated category [33, 1]. Examples are the cluster categories of finite type [15].

(6) The category of finitely generated projective modules over the ring $\mathbb{Z}/4\mathbb{Z}$ carries a triangulated structure that admits no model [42]; it is a locally finite triangulated category.

Orthogonal subcategories. Let T be a triangulated category and S a triangulated subcategory. Then we define two full subcategories

$$\mathsf{S}^\perp = \{Y \in \mathsf{T} \mid \mathrm{Hom}_\mathsf{T}(X, Y) = 0 \text{ for all } X \in \mathsf{S}\}$$

$$^\perp\mathsf{S} = \{X \in \mathsf{T} \mid \mathrm{Hom}_\mathsf{T}(X, Y) = 0 \text{ for all } Y \in \mathsf{S}\}$$

and call them *orthogonal subcategories* with respect to S. Note that $\mathsf{S}^\perp$ and $^\perp\mathsf{S}$ are thick subcategories of T.

The following lemma collects some basic facts about orthogonal subcategories which are well-known. For a proof, see [39, Proposition 4.9.1].

Lemma 2.4. *Let T be a triangulated category and S a thick subcategory. Then the following are equivalent.*

(1) *The inclusion functor $\mathsf{S} \to \mathsf{T}$ admits a right adjoint.*
(2) *The composite $\mathsf{S}^\perp \xrightarrow{\mathrm{inc}} \mathsf{T} \xrightarrow{\mathrm{can}} \mathsf{T}/\mathsf{S}$ is an equivalence.*
(3) *The inclusion functor $\mathsf{S}^\perp \to \mathsf{T}$ admits a left adjoint and $^\perp(\mathsf{S}^\perp) = \mathsf{S}$.* $\quad\square$

There is an interesting consequence. If S is a thick subcategory of T such that the inclusion admits a left and a right adjoint, then one has equivalences

$$^\perp\mathsf{S} \xrightarrow{\sim} \mathsf{T}/\mathsf{S} \xleftarrow{\sim} \mathsf{S}^\perp.$$

Existence of adjoints. Triangulated categories that are locally noetherian have the following remarkable property.

Theorem 2.5. *Let T be an essentially small triangulated category and suppose that T is locally noetherian. Then for each thick subcategory of T the inclusion functor admits a right adjoint.*

Proof. Fix a thick subcategory U and an object X in T. We need to construct a morphism $U \to X$ with U in U inducing a bijection $\mathrm{Hom}_{\mathsf{T}}(U', U) \to \mathrm{Hom}_{\mathsf{T}}(U', X)$ for all U' in U. Take the comma category U/X consisting of all morphisms $U \overset{\phi}{\to} X$ with U in U. A morphism from $U \overset{\phi}{\to} X$ to $U' \overset{\phi'}{\to} X$ is a morphism $\mu\colon U \to U'$ such that $\phi'\mu = \phi$. This category is closely related to Verdier's construction of the localisation functor $\mathsf{T} \to \mathsf{T}/\mathsf{U}$; see [48, II.2]. The arguments given there show that U/X is filtered. Therefore the functor

$$\underset{U \to X}{\mathrm{colim}}\, \mathrm{Hom}_{\mathsf{T}}(-, U)$$

is cohomological. Moreover, one obtains an exact sequence of cohomological functors

$$\cdots \longrightarrow \mathrm{Hom}_{\mathsf{T}/\mathsf{U}}(-, \Sigma^{-1}X) \longrightarrow \underset{U \to X}{\mathrm{colim}}\, \mathrm{Hom}_{\mathsf{T}}(-, U) \longrightarrow$$
$$\longrightarrow \mathrm{Hom}_{\mathsf{T}}(-, X) \longrightarrow \mathrm{Hom}_{\mathsf{T}/\mathsf{U}}(-, X) \longrightarrow \cdots$$

since one has by definition

$$\mathrm{Hom}_{\mathsf{T}/\mathsf{U}}(-, X) = \underset{X \to V}{\mathrm{colim}}\, \mathrm{Hom}_{\mathsf{T}}(-, V)$$

where $X \to V$ runs through all morphisms with cone in U.

Now we use that T is locally noetherian and write

$$\underset{U \to X}{\mathrm{colim}}\, \mathrm{Hom}_{\mathsf{T}}(-, U) = \bigoplus_{i \in I} \mathrm{Hom}_{\mathsf{T}}(-, U_i)$$

as a direct sum of representable functors. Similarly, we get

$$\mathrm{Hom}_{\mathsf{T}/\mathsf{U}}(-, X) = \bigoplus_{j \in J} \mathrm{Hom}_{\mathsf{T}}(-, V_j).$$

We may assume that U_i and V_j are non-zero for all i, j. Observe that $U_i \in \mathsf{U}$ and $V_j \in \mathsf{U}^{\perp}$ for all i, j. The morphism

$$\mathrm{Hom}_{\mathsf{T}}(-, X) \longrightarrow \bigoplus_{j \in J} \mathrm{Hom}_{\mathsf{T}}(-, V_j)$$

factors through a finite sum $\bigoplus_{j \in J_0} \mathrm{Hom}_{\mathsf{T}}(-, V_j)$. In fact, the exactness of the above sequence implies that V_j belongs to U for each $j \in J \smallsetminus J_0$. Thus $J = J_0$ and therefore $\mathrm{Hom}_{\mathsf{T}/\mathsf{U}}(-, X)$ belongs to $\mathbf{A}(\mathsf{T})$. It follows that I is also finite, since $\mathrm{colim}_{U \to X} \mathrm{Hom}_{\mathsf{T}}(-, U)$ is an extension of two objects in $\mathbf{A}(\mathsf{T})$. This yields a morphism $U = \coprod_i U_i \to X$ inducing a bijection $\mathrm{Hom}_{\mathsf{T}}(U', U) \to \mathrm{Hom}_{\mathsf{T}}(U', X)$ for all U' in U. $\qquad\square$

Corollary 2.6. *Let T be an essentially small triangulated category and suppose that T is locally noetherian. If U is a thick subcategory of T, then $^{\perp}(\mathsf{U}^{\perp}) = \mathsf{U}$.*

Proof. One could deduce this from Lemma 2.4, but we give the complete argument because it is short and simple. Clearly, $^{\perp}(\mathsf{U}^{\perp})$ contains U. Now pick an object X in $^{\perp}(\mathsf{U}^{\perp})$. Let $U \to X$ be the universal morphism from an object in U to X, and complete this to an exact triangle $U \to X \to V \to \Sigma U$. Then V belongs to $\mathsf{U}^{\perp}$ and therefore $\mathrm{Hom}_{\mathsf{T}}(X, V) = 0$. It follows that X belongs to U. $\qquad\square$

The following example shows that the identity $^{\perp}(\mathsf{U}^{\perp}) = \mathsf{U}$ does not hold in general.

Example 2.7. Consider the bounded derived category $\mathbf{D}^b(\operatorname{mod}\mathbb{Z})$ of finitely generated modules over the ring $\mathbb{Z}$ of integers. The complexes with torsion cohomology form a thick subcategory $\mathsf{U} = \mathbf{D}^b_{\mathrm{tor}}(\operatorname{mod}\mathbb{Z})$ such that $^\perp\mathsf{U} = 0$.

Corollary 2.8. *Let* T *be an essentially small triangulated category and suppose that* T *is locally noetherian. If* U *a thick subcategory of* T*, then the categories* U *and* T/U *are locally noetherian.*

Proof. The inclusion $\mathsf{U} \to \mathsf{T}$ induces a fully faithful and exact functor $\mathbf{A}(\mathsf{U}) \to \mathbf{A}(\mathsf{T})$. Thus every object of $\mathbf{A}(\mathsf{U})$ is noetherian. On the other hand, there is an equivalence $\mathsf{U}^\perp \xrightarrow{\sim} \mathsf{T}/\mathsf{U}$ by Lemma 2.4. Thus T/U is locally noetherian, since $\mathsf{U}^\perp$ is locally noetherian. $\square$

3. Auslander–Reiten theory for triangulated categories

We describe briefly the Auslander–Reiten theory for an essentially small triangulated category T that is locally noetherian. For general concepts from Auslander–Reiten theory we refer to Appendix A.

Auslander–Reiten triangles. Recall from [24] that an exact triangle

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

is an *Auslander–Reiten triangle* starting at X and ending at Z if α is a left almost split morphism and β is a right almost split morphism. Observe that these properties hold if and only if the morphism β is minimal right almost split, by [38, Lemma 2.6].

Proposition 3.1. *Given an essentially small triangulated category that is locally noetherian, there exists for each indecomposable object an Auslander–Reiten triangle ending at it.*

Proof. Fix an indecomposable object Z in T. Then the simple T-module $S_Z = \operatorname{Hom}_\mathsf{T}(-, Z)/\operatorname{Rad}_\mathsf{T}(-, Z)$ is finitely presented since T is locally noetherian. Thus we can choose in $\mathbf{A}(\mathsf{T})$ a minimal projective presentation

$$\operatorname{Hom}_\mathsf{T}(-, Y) \longrightarrow \operatorname{Hom}_\mathsf{T}(-, Z) \longrightarrow S_Z \longrightarrow 0;$$

see Proposition A.1. It follows from Lemma A.7 that the induced morphism $Y \to Z$ is minimal right almost split. Completing this morphism to an exact triangle yields an Auslander–Reiten triangle ending at Z. $\square$

The definition of an Auslander–Reiten triangle is symmetric. Thus there are Auslander–Reiten triangles in T starting at each indecomposable object if T^{op} is locally noetherian. This gives the existence of Auslander–Reiten triangles for locally finite triangulated categories. For compactly generated triangulated categories, this result is due to Beligiannis [11, Theorem 10.2].

Corollary 3.2. *Given an essentially small triangulated category that is locally finite, there exist Auslander–Reiten triangles starting and ending at each indecomposable object.* $\square$

Remark 3.3. Let $X \to Y \to Z \to \Sigma X$ be an Auslander–Reiten triangle in an essentially small triangulated category that is locally noetherian. The relation between the end terms can be explained as follows. Let $A = \operatorname{End}_\mathsf{T}(Z)$ and denote by $E = E(A/\operatorname{rad} A)$ an injective envelope. Then

$$(3.1) \qquad \operatorname{Hom}_A(\operatorname{Hom}_\mathsf{T}(Z, -), E) \cong \operatorname{Hom}_\mathsf{T}(-, \Sigma X);$$

see [38, Theorem 2.2]. In particular,

$$\begin{aligned}
\mathrm{End}_{\mathsf{T}}(X) &\cong \mathrm{End}_{\mathsf{T}}(\Sigma X) \\
&\cong \mathrm{Hom}_A(\mathrm{Hom}_{\mathsf{T}}(Z, \Sigma X), E) \\
&\cong \mathrm{Hom}_A(\mathrm{Hom}_A(\mathrm{Hom}_{\mathsf{T}}(Z, Z), E), E) \\
&\cong \mathrm{End}_A(E).
\end{aligned}$$

The Auslander–Reiten quiver. For a locally finite triangulated category the structure of the Auslander–Reiten quiver has been determined in work of Xiao and Zhu [51] and Amiot [1]. In fact, these authors consider triangulated categories that are linear over a field with finite dimensional morphism spaces. Then they apply the structural results on valued translation quivers due to Riedtmann [43] and Happel, Preiser, and Ringel [26]. The same arguments work in a slightly more general setting, thanks to the following result; see also [41, Proposition 2.1].

For the definition of a *valued translation quiver*, see [41, §2]. The original definition [27, §2] excludes loops, but they are possible in our setting.

Proposition 3.4. *Let k be a commutative ring and T an essentially small k-linear triangulated category such that all morphism spaces are of finite length over k. Suppose that T is locally finite. Then the Auslander–Reiten quiver of T is a valued translation quiver. Assigning to a vertex X the length $\ell(X)$ of $\mathrm{Hom}_{\mathsf{T}}(-, X)$ in the abelianisation $\mathbf{A}(\mathsf{T})$ yields a subadditive function on the set of vertices such that for each vertex Z*

$$2\ell(Z) = \ell(Z) + \ell(\tau Z) = 2 + \sum_{Y \to Z} \delta_{Y,Z}\ell(Y).$$

Proof. The existence of Auslander–Reiten triangles has already been established, and this gives the translation τ. The identities for the valuation are precisely the statements of Lemmas A.11 and A.12. For the second part one uses the fact that each Auslander–Reiten triangle $\tau Z \to \bar{Y} \to Z \to \Sigma(\tau Z)$ induces an exact sequence

$$0 \to S_{\Sigma^{-1}Z} \to \mathrm{Hom}_{\mathsf{T}}(-, \tau Z) \to \mathrm{Hom}_{\mathsf{T}}(-, \bar{Y}) \to \mathrm{Hom}_{\mathsf{T}}(-, Z) \to S_Z \to 0$$

in $\mathbf{A}(\mathsf{T})$ by Lemma A.7. Then one applies Lemma A.8 which gives a decomposition

$$\bar{Y} = \coprod_{Y \to Z} Y^{\delta_{Y,Z}}$$

where $Y \to Z$ runs through all arrows ending at Z. It remains to observe that $\ell(Z) = \ell(\tau Z)$. This follows from the isomorphism (3.1) and the alternative description of ℓ via

$$\ell(X) = \sum_C \ell_{\mathrm{End}_{\mathsf{T}}(C)} \mathrm{Hom}_{\mathsf{T}}(C, X),$$

where C runs through a representative set of indecomposable objects; see [3, Proposition 2.11]. Note that $\mathrm{Hom}_{\mathsf{T}}(-, X)$ and $\mathrm{Hom}_{\mathsf{T}}(X, -)$ have the same length, thanks to the duality (2.1). $\qquad\square$

Theorem 3.5 (Xiao–Zhu). *Let k be a commutative ring and T an essentially small k-linear triangulated category such that all morphism spaces are of finite length over k. Suppose that T is locally finite. Then each connected component of the Auslander–Reiten quiver of T is of the form $\mathbb{Z}\Delta/G$ for some tree Δ of Dynkin type and some group G of automorphisms of $\mathbb{Z}\Delta$.*

Proof. Adapt the proofs of [51, Theorem 2.3.4] or [1, Theorem 4.0.4], using Proposition 3.4. $\qquad\square$

Example 3.6. Let k be a field and Γ be a quiver of Dynkin type. Then the Auslander–Reiten quiver of $\mathbf{D}^b(\operatorname{mod} k\Gamma)$ is of the form $\mathbb{Z}\Gamma$; see [24, §4].

A *decomposition* $\mathsf{C} = \mathsf{C}_1 \times \mathsf{C}_2$ of an additive category C is a pair of full additive subcategories C_1 and C_2 such that each object in C is a direct sum of two objects from C_1 and C_2, and $\operatorname{Hom}_{\mathsf{C}}(X_1, X_2) = 0 = \operatorname{Hom}_{\mathsf{C}}(X_2, X_1)$ for all $X_1 \in \mathsf{C}_1$ and $X_2 \in \mathsf{C}_2$. An additive category C is *connected* if any decomposition $\mathsf{C} = \mathsf{C}_1 \times \mathsf{C}_2$ implies $\mathsf{C} = \mathsf{C}_1$ or $\mathsf{C} = \mathsf{C}_2$.

Proposition 3.7. *Let T be an essentially small triangulated category that is locally finite. Then any non-zero morphism $X \to Y$ between two indecomposable objects in T gives a path $X \to \cdots \to Y$ in the Auslander–Reiten quiver of T. In particular, the category T is connected if and only if its Auslander–Reiten quiver is connected.*

Proof. We prove the first statement; the second statement is then an immediate consequence. Let $\phi \colon X \to Y$ be a non-zero morphism. If ϕ is invertible, then the path between X and Y has length zero. Otherwise, ϕ factors through the right almost split morphism ending at $Y_0 = Y$, which exists by Proposition 3.1. It follows from Lemma A.8 that there is an arrow $Y_1 \to Y_0$ and a non-zero morphism $X \to Y_0$ that factors through an irreducible morphism $Y_1 \to Y_0$. We continue with the corresponding morphism $X \to Y_1$, and the process terminates since T is locally finite. $\qquad\square$

It is interesting to note that the Auslander–Reiten quiver of T can be identified with the Ext-quiver [22, 7.1] of the abelian length category $\mathbf{A}(\mathsf{T})$, using the bijection from Lemma A.5 between the indecomposable objects of T and the simple objects of $\mathbf{A}(\mathsf{T})$. For the bijection between arrows one uses Lemma A.6.

The Nakayama functor. Let k be a field and T an essentially small k-linear triangulated category with finite dimensional morphism spaces. Suppose that T is locally finite. Then there is for each object X in T an object NX representing the k-dual of $\operatorname{Hom}_{\mathsf{T}}(X, -)$, that is,

$$\operatorname{Hom}_k(\operatorname{Hom}_{\mathsf{T}}(X, -), k) \cong \operatorname{Hom}_{\mathsf{T}}(-, NX).$$

More precisely, we have an isomorphism

$$\operatorname{Hom}_k(\operatorname{Hom}_{\mathsf{T}}(X, -), k) \cong \bigoplus_{i \in I} \operatorname{Hom}_{\mathsf{T}}(-, Y_i)$$

for some collection of indecomposable objects Y_i since T is locally noetherian. It follows from Proposition 2.3 that I is finite. Thus $NX = \coprod_i Y_i$ is a representing object. This gives a functor $N \colon \mathsf{T} \to \mathsf{T}$ which is known as *Nakayama functor* in representation theory [23], or as *Serre functor* in algebraic geometry [13]. It is easily checked that this functor is an equivalence; a quasi-inverse is given by the Nakayama functor for T^{op} which sends an object X to the object representing $\operatorname{Hom}_k(\operatorname{Hom}_{\mathsf{T}}(-, X), k)$. The exactness then follows from [13, Proposition 3.3] or [47, Theorem A.4.4].

Note that for each indecomposable object Z in T, one obtains an Auslander–Reiten triangle $X \to Y \to Z \to \Sigma X$ by first choosing a non-zero k-linear map $\operatorname{End}_{\mathsf{T}}(Z) \to k$ annihilating the unique maximal ideal of $\operatorname{End}_{\mathsf{T}}(Z)$ and then completing the corresponding morphism $Z \to NZ$ to an exact triangle $\Sigma^{-1}(NZ) \to Y \to Z \to NZ$; see [38] for details.

4. The lattice of thick subcategories

Let T be an essentially small triangulated category. We denote by $\mathbf{T}(\mathsf{T})$ the set of all thick subcategories of T. This set is partially ordered by inclusion. Observe that for any collection of thick subcategories U_i its intersection $\bigcap_i \mathsf{U}_i$ is thick. Thus $\mathbf{T}(\mathsf{T})$ is a *lattice*, that is, we can form for each pair of thick subcategories U, V its infimum $\mathsf{U} \wedge \mathsf{V} = \mathsf{U} \cap \mathsf{V}$ and its supremum $\mathsf{U} \vee \mathsf{V} = \bigcap_{\mathsf{S}} \mathsf{S}$, where S runs through all thick subcategories containing U and V. In fact, the lattice $\mathbf{T}(\mathsf{T})$ is *complete* since for any collection of thick subcategories U_i its infimum $\bigwedge_i \mathsf{U}_i$ and its supremum $\bigvee_i \mathsf{U}_i$ exist.

If X is an object or collection of objects in T, we write $\mathrm{Thick}(X)$ for the thick subcategory generated by X, that is, the smallest thick subcategory containing T. The triangulated category T is *finitely generated* if there is some object X such that $\mathsf{T} = \mathrm{Thick}(X)$.

Compactness. Let L be a lattice. An element x of L is *compact* if $x \leq \bigvee_{i \in I} y_i$ implies $x \leq \bigvee_{i \in J} y_i$ for some finite subset $J \subseteq I$. The lattice L is *compact* if it has a greatest element that is compact.

Lemma 4.1. *The lattice $\mathbf{T}(\mathsf{T})$ is compact if and only if T is finitely generated.* $\qquad\square$

Noetherianess. A lattice L is *noetherian* if there is no infinite strictly increasing chain $x_0 < x_1 < x_2 < \ldots$ in L.

Proposition 4.2. *Let T be an essentially small triangulated category that is finitely generated and locally noetherian. Then the lattice $\mathbf{T}(\mathsf{T})$ is noetherian.*

Proof. Let $\mathsf{T} = \mathrm{Thick}(X)$ and write $H = \mathrm{Hom}_{\mathsf{T}}(-, X)$. For each thick subcategory U let $\phi_{\mathsf{U}} \colon X_{\mathsf{U}} \to X$ be the universal morphism from an object in U ending at X, which exists by Theorem 2.5. Note that $\mathsf{U} = \mathrm{Thick}(X_{\mathsf{U}})$ since $\mathsf{T} = \mathrm{Thick}(X)$. Denote by H_{U} the image of the induced morphism $\mathrm{Hom}_{\mathsf{T}}(-, X_{\mathsf{U}}) \to H$ in $\mathbf{A}(\mathsf{T})$, and observe that the induced morphism $\pi_{\mathsf{U}} \colon \mathrm{Hom}_{\mathsf{T}}(-, X_{\mathsf{U}}) \to H_{\mathsf{U}}$ is a projective cover. This follows from the uniqueness of ϕ_{U} with Lemma A.2, since any endomorphism $\mathrm{Hom}_{\mathsf{T}}(-, X_{\mathsf{U}}) \to \mathrm{Hom}_{\mathsf{T}}(-, X_{\mathsf{U}})$ commuting with π_{U} is an isomorphism.

Let V be another thick subcategory of T. Then $H_{\mathsf{U}} = H_{\mathsf{V}}$ implies $\mathsf{U} = \mathsf{V}$. Indeed, $H_{\mathsf{U}} = H_{\mathsf{V}}$ implies that their projective covers are isomorphic. Thus $X_{\mathsf{U}} \cong X_{\mathsf{V}}$, and therefore

$$\mathsf{U} = \mathrm{Thick}(X_{\mathsf{U}}) = \mathrm{Thick}(X_{\mathsf{V}}) = \mathsf{V}.$$

Clearly, $\mathsf{U} \subseteq \mathsf{V}$ implies $H_{\mathsf{U}} \subseteq H_{\mathsf{V}}$. It follows that $\mathbf{T}(\mathsf{T})$ is noetherian, since the lattice of subobjects of H in $\mathbf{A}(\mathsf{T})$ is noetherian. $\qquad\square$

Example 4.3. Recall from Example 2.2 that representations of the quiver

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow \cdots$$

give rise to a triangulated category $\mathbf{D}^b(\mathsf{A})$ that is locally noetherian. The full subcategories $\mathsf{A}_n \subseteq \mathsf{A}$ consisting of all representations with support contained in $\{1, \ldots, n\}$ yield an infinite strictly increasing chain $\mathbf{D}^b(\mathsf{A}_1) \subseteq \mathbf{D}^b(\mathsf{A}_2) \subseteq \ldots$ of thick subcategories in $\mathbf{D}^b(\mathsf{A})$.

Complements. Assigning to a thick subcategory U its orthogonal subcategories $\mathsf{U}^{\perp}$ and $^{\perp}\mathsf{U}$ yields two order reversing maps $\mathbf{T}(\mathsf{T}) \to \mathbf{T}(\mathsf{T})$. These maps are of interest because we have for two objects X, Y in T

$$\mathrm{Hom}_{\mathsf{T}}^*(X, Y) = 0 \quad \Longleftrightarrow \quad \mathrm{Thick}(X)^{\perp} \geq \mathrm{Thick}(Y)$$

where
$$\mathrm{Hom}^*_\mathsf{T}(X,Y) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_\mathsf{T}(X, \Sigma^n Y).$$

Let us collect the basic properties of both maps when T is locally finite.

Proposition 4.4. *Let T be an essentially small triangulated category and suppose that T is locally finite.*

  (1) *The maps $\mathbf{T}(\mathsf{T}) \to \mathbf{T}(\mathsf{T})$ taking U to $\mathsf{U}^\perp$ and $^\perp\mathsf{U}$ are mutually inverse. Thus the lattice $\mathbf{T}(\mathsf{T})$ is self-dual.*
  (2) *Let U be a thick subcategory. Then $\mathsf{U} \wedge \mathsf{U}^\perp = 0$ and $\mathsf{U} \vee \mathsf{U}^\perp = \mathsf{T}$.*
  (3) *Let $\mathsf{V} \subseteq \mathsf{U} \subseteq \mathsf{T}$ be thick subcategories. Then the quotient U/V is a locally finite triangulated category, and there is a lattice isomorphism*
$$[\mathsf{V},\mathsf{U}] = \{\mathsf{X} \in \mathbf{T}(\mathsf{T}) \mid \mathsf{V} \subseteq \mathsf{X} \subseteq \mathsf{U}\} \xrightarrow{\sim} \mathbf{T}(\mathsf{U}/\mathsf{V}).$$
  The map takes $\mathsf{X} \subseteq \mathsf{U}$ to X/V.

Proof. (1) We have $^\perp(\mathsf{U}^\perp) = \mathsf{U} = (^\perp\mathsf{U})^\perp$ by Corollary 2.6.

(2) Clearly, $\mathsf{U} \cap \mathsf{U}^\perp = 0$. On the other hand, each object X in T fits into an exact triangle $U \to X \to V \to \Sigma U$ with $U \in \mathsf{U}$ and $V \in \mathsf{U}^\perp$, by Theorem 2.5. Thus $\mathsf{U} \vee \mathsf{U}^\perp = \mathsf{T}$.

(3) The category U/V is locally finite by Corollary 2.8, and the inverse map $\mathbf{T}(\mathsf{U}/\mathsf{V}) \to [\mathsf{V},\mathsf{U}]$ takes X to its inverse image under the localisation functor $\mathsf{U} \to \mathsf{U}/\mathsf{V}$. $\qquad\square$

This proposition says that the lattice is *relatively complemented*, that is, each interval is complemented. A lattice L is *complemented* if for each $x \in L$ there exists $y \in L$ such that $x \vee y = 1$ and $x \wedge y = 0$.

If T is locally finite and admits a Nakayama functor $N\colon \mathsf{T} \xrightarrow{\sim} \mathsf{T}$, then this induces a lattice automorphism of $\mathbf{T}(\mathsf{T})$ by taking a thick subcategory U to $N\mathsf{U}$. It follows from the definition of N that $^\perp N\mathsf{U} = \mathsf{U}^\perp$. Thus $N\mathsf{U} = (\mathsf{U}^\perp)^\perp$ by Corollary 2.6.

Finiteness. There are further finiteness results for the lattice $\mathbf{T}(\mathsf{T})$ if T is locally finite.

Proposition 4.5. *Let T be an essentially small triangulated category. If T is finitely generated and locally finite, then T has only finitely many thick subcategories.*

Proof. Let $\mathsf{T} = \mathrm{Thick}(X)$. For each thick subcategory U let $X_\mathsf{U} \to X$ be the universal morphism from an object in U ending at X, which exists by Theorem 2.5. Note that $\mathsf{U} = \mathrm{Thick}(X_\mathsf{U})$ since $\mathsf{T} = \mathrm{Thick}(X)$. Each indecomposable direct summand X' of X_U satisfies $\mathrm{Hom}_\mathsf{T}(X', X) \neq 0$. There are only finitely many isomorphism classes of such objects by Proposition 2.3. It follows that $\mathbf{T}(\mathsf{T})$ is finite. $\qquad\square$

Auslander–Reiten theory is used in an essential way for proving the following.

Theorem 4.6. *Let T be an essentially small and locally finite triangulated category. If T is connected then T is finitely generated and has therefore only finitely many thick subcategories.*

We need the following lemma.

Lemma 4.7 (Amiot)**.** *Let T be an essentially small triangulated category that is locally finite. Then any connected component of the Auslander–Reiten quiver of T is – after removing possible loops – of the form $\mathbb{Z}\Delta/G$ for some finite tree Δ and some group G of automorphisms of $\mathbb{Z}\Delta$.*

Proof. If $X \to X$ is an irreducible morphism in T then $X = \tau X$, by [1, Proposition 4.1.1]. After removing such arrows $X \to X$, the Auslander–Reiten quiver of T is a stable translation quiver and therefore of the form $\mathbb{Z}\Delta/G$ by Riedtmann's Struktursatz [43]. The tree is finite by [1, Lemma 4.2.2]. $\square$

Proof of Theorem 4.6. The Auslander–Reiten quiver of T is connected by Proposition 3.7 and therefore – modulo loops – of the form $\mathbb{Z}\Delta/G$ for some finite tree Δ and some group G of automorphisms by Lemma 4.7. Note that $\mathbb{Z}\Delta$ does not depend on the orientation of Δ. Thus we may assume that Δ has no path of length > 1. Then there are two types of vertices. Call a vertex $x \in \Delta_0$ *minimal* if there is no arrow from Δ ending at x, and *maximal* if there is no arrow from Δ starting at x.

Let $\pi \colon \mathbb{Z}\Delta \to \mathbb{Z}\Delta/G$ be the canonical map and write each vertex of $\mathbb{Z}\Delta$ as $\tau^r x$ with $r \in \mathbb{Z}$ and $x \in \Delta_0$. We claim that $\mathsf{T} = \mathrm{Thick}(T)$ for $T = \coprod_{x \in \Delta_0} \pi(x)$. Thus we fix an indecomposable object $X = \pi(y)$ in T and need to show that X belongs to $\mathrm{Thick}(T)$. Define for $y = \tau^r x$

$$d(y) = \begin{cases} 2r, & \text{if } x \text{ is maximal}, \\ 2r - 1, & \text{if } x \text{ is minimal}, \end{cases}$$

and use induction on $d = d(y)$ as follows. The cases $d = 0$ and $d = -1$ are clear. Suppose now $d > 0$ and consider the Auslander–Reiten triangle $X \to Y \to \tau^{-1}X \to \Sigma X$ starting at X. Then each indecomposable direct summand of Y is of the form $\pi(y')$ with an arrow $y \to y'$ in $\mathbb{Z}\Delta$, by Lemma A.8. We have $d(y') = d - 1$ and $d(\tau^{-1}y) = d - 2$. Thus Y and $\tau^{-1}X$ belong to $\mathrm{Thick}(T)$, and it follows that X is in $\mathrm{Thick}(T)$. The case $d < 0$ is similar.

The finiteness of $\mathbf{T}(\mathsf{T})$ follows from Proposition 4.5. $\square$

5. Simply connected triangulated categories

Let T be a triangulated category that is essentially small and locally finite. We say that T is *simply connected* if the Auslander–Reiten quiver of T is connected and has no oriented cycle. Here, an *oriented cycle* is a path of length > 0 starting and ending at the same vertex. Note that there exists an oriented cycle if and only if there is a chain of non-invertible non-zero morphisms $X_0 \to X_1 \to \cdots \to X_n = X_0$ between indecomposable objects; see Proposition 3.7.

Rings of finite representation type. For any ring A, we denote by $\mathrm{mod}\, A$ the category of finitely presented A-modules. Recall that A has *finite representation type* if A is artinian and there are only finitely many isomorphism classes of finitely presented indecomposable A-modules.

Theorem 5.1. *For a triangulated category T the following are equivalent.*

(1) *The triangulated category T is essentially small, algebraic, locally finite, and simply connected.*
(2) *There exists a connected hereditary artinian ring A of finite representation type such that T is equivalent to the bounded derived category $\mathbf{D}^b(\mathrm{mod}\, A)$.*

In this case the Auslander–Reiten quiver of T is of the form $\mathbb{Z}\Delta$ for some tree Δ of Dynkin type.

Let us explain how the ring A is obtained from T. A *species* $(K_i, {}_iE_j)_{i,j \in I}$ consists of a family of division rings K_i and a family of $K_i - K_j$-bimodules ${}_iE_j$.

There is an associated tensor algebra $\bigoplus_{n \geq 0} M^n$, where

$$M^0 = \prod_{i \in I} K_i, \quad M^1 = \bigoplus_{i,j \in I} {}_i E_j, \quad M^n = M^1 \otimes_{M^0} \ldots \otimes_{M^0} M^1 \quad \text{for} \quad n > 1,$$

and multiplication is induced by the tensor product.

Let T be a triangulated category and suppose that T is locally finite and simply connected. Then the Auslander–Reiten quiver is of the form $\mathbb{Z}\Delta$ for some finite tree by Lemma 4.7. In fact, we may assume that Δ has no path of length > 1, since $\mathbb{Z}\Delta$ does not depend on the orientation of Δ. Choose indecomposable objects T_i for each vertex $i \in \Delta_0$. Define $K_i = \operatorname{End}_{\mathsf{T}}(T_i)$ and ${}_i E_j = \operatorname{Hom}_{\mathsf{T}}(T_j, T_i)$ for $i, j \in \Delta_0$. Observe that each K_i is a division ring by Proposition 3.7, and that each bimodule ${}_i E_j$ is of finite length on either side by Proposition 2.3. Then the tensor algebra corresponding to the species $(K_i, {}_i E_j)_{i,j \in \Delta_0}$ is isomorphic to the endomorphism ring of $T = \coprod_{i \in \Delta_0} T_i$. The proof is straightforward, using the fact that there are no paths of length > 1 in Δ. We denote this ring by A. Observe that A is hereditary and artinian, since $K = \prod_{i \in \Delta_0} K_i$ is semisimple and $\bigoplus_{i,j \in \Delta_0} {}_i E_j$ is finitely generated over K on either side.

Lemma 5.2. *The object T is a tilting object, that is, T is generated by T and $\operatorname{Hom}_{\mathsf{T}}(T, \Sigma^n T) = 0$ for all $n \neq 0$.*

Proof. We use Proposition 3.7 which gives a path in the Auslander–Reiten quiver for each non-zero morphism between indecomposable objects. In particular, there is for each indecomposable object X a path $X \to Y \to \tau^{-1}X \to \cdots \to \Sigma X$, provided there is a non-zero morphism starting in X which is not a section.

Suppose that $\operatorname{Hom}_{\mathsf{T}}(T_i, \Sigma^n T_j) \neq 0$ and consider the following two cases:

$n > 0$. From Remark 3.3 one has $\operatorname{Hom}_{\mathsf{T}}(\Sigma^{n-1} T_j, \tau T_i) \neq 0$. Thus there is in $\mathbb{Z}\Delta$ a path

$$T_j \to \cdots \to \Sigma^{n-1} T_j \to \cdots \to \tau T_i \to T_i' \to T_i.$$

$n < 0$. There is in $\mathbb{Z}\Delta$ a path

$$T_i \to \cdots \to \Sigma^n T_j \to \cdots \to \tau T_j \to T_j' \to T_j.$$

In both cases, this contradicts the choice of $T_i, T_j \in \Delta_0$ and the fact that Δ has no path of length > 1. It follows that $n = 0$.

The fact that $\mathsf{T} = \operatorname{Thick}(T)$ follows from the proof of Theorem 4.6. $\square$

Proof of Theorem 5.1. (1) $\Rightarrow$ (2): The assumptions on T yield a tilting object $T = \coprod_{i \in \Delta_0} T_i$, and its endomorphism ring $A = \operatorname{End}_{\mathsf{T}}(T)$ is hereditary artinian. It follows that there are equivalences

$$\mathsf{T} \xleftarrow{\sim} \mathbf{K}^b(\operatorname{add} T) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod} A).$$

For the first equivalence, see [32, 2.1], while the second is clear from the fact that $\operatorname{Hom}_{\mathsf{T}}(T, -)$ identifies $\operatorname{add} T$ with the category of finitely generated projective A-modules. Locally finiteness of T implies that A is of finite representation type.

(2) $\Rightarrow$ (1): The abelian category $\operatorname{mod} A$ is hereditary, that is, $\operatorname{Ext}_A^p(-,-)$ vanishes for $p > 1$. It follows that each indecomposable complex is concentrated in a single degree. Finite representation type of A implies that T is locally finite, by Proposition 2.3. The Auslander–Reiten triangles in $\mathbf{D}^b(\operatorname{mod} A)$ are obtained from almost split seqences in $\operatorname{mod} A$. Thus the Auslander–Reiten quiver is of the form $\mathbb{Z}\Delta$ for some tree Δ; see [24, §4] for details. In particular, the quiver has no oriented cycles.

For the shape of Δ, see [18, §4], where hereditary rings of finite representation type are discussed. $\square$

6. Thick subcategories and non-crossing partitions

Let A be a *hereditary Artin algebra*. Thus A is an artinian ring that is finitely generated over its centre and $\operatorname{Ext}_A^p(-,-) = 0$ for all $p > 1$. Note that the centre of a hereditary Artin algebra is semisimple. The algebra A is said to be *connected* if the centre of A is a field. From now on assume that A is connected and we denote the centre by k.

In this section, we associate to A a poset of non-crossing partitions and establish a correspondence between this poset and the lattice of thick subcatgories of $\mathbf{D}^b(\operatorname{mod} A)$. This correspondence was first observed for path algebras of finite and affine type by Ingalls and Thomas [29]. For an introduction to non-crossing partitions, see [2].

The Weyl group. Fix a representative set $S_1, \ldots, S_n$ of simple A-modules. We associate to A a generalised Cartan matrix $C(A) = (C_{ij})_{1 \le i,j \le n}$ as follows. Given two simple modules S_i, S_j, we have $\operatorname{Ext}_A^1(S_i, S_j) = 0$ or $\operatorname{Ext}_A^1(S_j, S_i) = 0$. Assume $i \ne j$ and $\operatorname{Ext}_A^1(S_j, S_i) = 0$. Then define

$$C_{ij} = -\ell_{\operatorname{End}_A(S_i)}(\operatorname{Ext}_A^1(S_i, S_j)) \quad \text{and} \quad C_{ji} = -\ell_{\operatorname{End}_A(S_j)}(\operatorname{Ext}_A^1(S_i, S_j)).$$

In addition, define $C_{ii} = 2$ and $d_i = \ell_k(\operatorname{End}_A(S_i))$ for each i. Then one has $d_i C_{ij} = d_j C_{ji}$. Thus $C(A)$ is a symmetrisable generalised Cartan matrix in the sense of [31].

Next we consider the Weyl group corresponding to $C(A)$. Let $\mathbb{R}^n$ denote the n-dimensional real space with standard basis $\varepsilon_1, \ldots, \varepsilon_n$. Define a symmetric bilinear form by $(\varepsilon_i, \varepsilon_j) = d_i C_{ij}$ and for each $\alpha \in \mathbb{R}^n$ with $(\alpha, \alpha) \ne 0$ the reflection

$$s_\alpha \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad \xi \mapsto \xi - 2\frac{(\xi, \alpha)}{(\alpha, \alpha)}\alpha.$$

We write s_i for the *simple reflection* s_{ε_i} and observe that each s_i maps $\mathbb{Z}^n$ into itself. The *Weyl group* is the group W generated by the simple reflections $s_1, \ldots, s_n$. The *real roots* are by definition the elements of $\mathbb{Z}^n$ of the form $w(\varepsilon_i)$ for some $w \in W$ and some $i \in \{1, \ldots, n\}$. Note that for any real root α the corresponding reflection s_α belongs to W since $s_{w(\alpha)} = w s_\alpha w^{-1}$.

We define the *absolute order* on W with respect to the *absolute length* ℓ as follows. Consider the set of reflections

$$W_1 = \{w s_i w^{-1} \mid w \in W, 1 \le i \le n\}$$

and for each $w \in W$ let $\ell(w)$ denote the minimal $r \ge 0$ such that w can be written as product $w = x_1 \ldots x_r$ of reflections $x_j \in W_1$. Given $u, v \in W$ define

$$u \le v \quad \Longleftrightarrow \quad \ell(u) + \ell(u^{-1}v) = \ell(v).$$

Note that the length function ℓ and the absolute order are invariant under conjugation with a fixed element of W.

A *Coxeter element* in W is any element of W that is conjugate to one of the form $s_{\sigma(1)} s_{\sigma(2)} \ldots s_{\sigma(n)}$ for some permutation σ. Note that $\ell(c) = n$ for each Coxeter element c by [19, Theorem 1.1]. This has the following immediate consequence.

Lemma 6.1. *Let c be a Coxeter element and $x_1, \ldots, x_n$ a sequence of reflections in W_1 such that $c = x_1 \cdots x_n$. If $1 \le r \le s \le n$, then*

$$\ell(x_1 \ldots x_r) = r \quad \text{and} \quad x_1 \ldots x_r \le x_1 \ldots x_s. \qquad \square$$

Relative to a Coxeter element c one defines the poset of *non-crossing partitions*

$$\text{NC}(W, c) = \{w \in W \mid \text{id} \leq w \leq c\}.$$

Given two Coxeter elements c, c' in W, we have $\text{NC}(W, c) \cong \text{NC}(W, c')$ provided that c and c' are conjugate.

The Grothendieck group $K_0(\text{mod } A)$ is isomorphic to $\mathbb{Z}^n$ via the map sending each simple A-module S_i to the standard base vector ε_i. The image of an A-module X under this map is called *dimension vector* and denoted by $\dim X$; we write $s_X = s_{\dim X}$ for the corresponding reflection.

Exceptional modules and sequences. An A-module X is called *exceptional* if X is indecomposable and $\text{Ext}_A^1(X, X) = 0$. Note that $\dim X$ is a real root if X is exceptional [45, Corollary 2]. A sequence $(X_1, \ldots, X_r)$ of A-modules is called *exceptional* if each X_i is exceptional and

$$\text{Hom}_A(X_j, X_i) = 0 = \text{Ext}_A^1(X_j, X_i) \quad \text{for all} \quad i < j.$$

Such a sequence is *complete* if $r = n$.

Lemma 6.2. *Sending an A-module X to the reflection $s_X = s_{\dim X}$ gives an injective map from the set of isomorphism classes of exceptional A-modules into W.*

Proof. An exceptional A-module is uniquely determined by its dimension vector; see [35, Lemma 8.2]. On the other hand, given a reflection $s \in W_1$, we have $s = s_\alpha$ where $\alpha \in \mathbb{Z}^n$ is the unique vector with non-negative entries satisfying $s(\alpha) = -\alpha$. Thus $s_{\dim X} = s_{\dim Y}$ implies $X \cong Y$. $\square$

Theorem 6.3 (Crawley-Boevey, Ringel, Igusa–Schiffler). *Let A be a connected hereditary Artin algebra with simple modules $S_1, \ldots, S_n$ satisfying $\text{Ext}_A^1(S_j, S_i) = 0$ for all $i < j$. Denote by W the associated Weyl group and fix the Coxeter element $c = s_1 \cdots s_n$. Then the braid group B_n on n strings acts transitively on*

— *the isomorphism classes of complete exceptional sequences $(X_1, \ldots, X_n)$ in* $\text{mod } A$, *and*
— *the sequences $(x_1, \ldots, x_n)$ of reflections in W_1 such that $c = x_1 \cdots x_n$.*

Moreover, $\sigma(X_1, \ldots, X_n) = (Y_1, \ldots, Y_n)$ implies $\sigma(s_{X_1}, \ldots, s_{X_n}) = (s_{Y_1}, \ldots, s_{Y_n})$ for all $\sigma \in B_n$.

Proof. For the action of the braid group on complete exceptional sequences, see [17, 45]. For the action on factorisations of the Coxeter element, see [28, Theorem 1.4]. The compatibility of both actions follows from the computation of the dimension vectors of the modules in an exceptional sequence under the braid group action; see the proof of [28, Corollary 2.4] and also the Corollary in [17]. $\square$

Corollary 6.4. *Let $(x_1, \ldots, x_n)$ be a sequence of reflections in W_1 such that $c = x_1 \cdots x_n$. Then there exists up to isomorphism a unique complete exceptional sequence $(X_1, \ldots, X_n)$ such that $x_i = s_{X_i}$ for all i.*

Proof. Theorem 6.3 gives $\sigma \in B_n$ such that $(x_1, \ldots, x_n) = \sigma(s_1, \ldots, s_n)$. Let $(X_1, \ldots, X_n) = \sigma(S_1, \ldots, S_n)$. Then $x_i = s_{X_i}$ for all i. Uniqueness follows from Lemma 6.2. $\square$

Thick subcategories. We consider the bounded derived category $\mathbf{D}^b(\operatorname{mod} A)$ and identify A-modules with complexes concentrated in degree zero. Recall the following correspondence.

Proposition 6.5 (Brüning)**.** *Let* A *be a hereditary abelian category. The canonical inclusion* $\mathsf{A} \to \mathbf{D}^b(\mathsf{A})$ *induces a bijection between*

  – *the set of exact abelian and extension closed subcategories of* A, *and*
  – *the set of thick subcategories of* $\mathbf{D}^b(\mathsf{A})$.

The bijection sends $\mathsf{C} \subseteq \mathsf{A}$ *to* $\{X \in \mathbf{D}^b(\mathsf{A}) \mid H^i X \in \mathsf{C}$ *for all* $i \in \mathbb{Z}\}$. *Its inverse sends* $\mathsf{U} \subseteq \mathbf{D}^b(\mathsf{A})$ *to*

$$H^0 \mathsf{U} = \{Y \in \mathsf{A} \mid Y \cong H^0 X \text{ for some } X \in \mathsf{U}\}.$$

Proof. See [14, Theorem 5.1]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The *Loewy length* of an object X in some abelian category is the smallest $p \geq 0$ such that there exists a chain $0 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_p = X$ so that X_{i+1}/X_i is semisimple for all i. The *height* of an abelian category is the supremum of the Loewy lengths of its objects.

Next we characterise the thick subcategories of $\mathbf{D}^b(\operatorname{mod} A)$ such that the inclusion admits an adjoint. The connection with exceptional sequences is due to Bondal [12].

Proposition 6.6. *Let A be a hereditary Artin algebra. For a thick subcategory* U *of* $\mathbf{D}^b(\operatorname{mod} A)$ *are equivalent.*

  (1) *The inclusion* $\mathsf{U} \to \mathbf{D}^b(\operatorname{mod} A)$ *admits a left adjoint.*
  (2) *The inclusion* $H^0\mathsf{U} \to \operatorname{mod} A$ *admits a left adjoint.*
  (3) *The abelian category* $H^0\mathsf{U}$ *is of finite height and has only finitely many isomorphism classes of simple objects.*
  (4) *There exists an exceptional sequence* $(X_1, \ldots, X_r)$ *in* $\operatorname{mod} A$ *such that* $\mathsf{U} = \operatorname{Thick}(X_1, \ldots, X_r)$.
  (5) *There exists a complete exceptional sequence* $(X_1, \ldots, X_n)$ *in* $\operatorname{mod} A$ *such that*

$$\mathsf{U} = \operatorname{Thick}(X_1, \ldots, X_r) \quad and \quad {}^\perp \mathsf{U} = \operatorname{Thick}(X_{r+1}, \ldots, X_n)$$

Remark 6.7. The k-duality $(\operatorname{mod} A)^{\operatorname{op}} \xrightarrow{\sim} \operatorname{mod}(A^{\operatorname{op}})$ induces a duality

$$\mathbf{D}^b(\operatorname{mod} A)^{\operatorname{op}} \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod} A^{\operatorname{op}})$$

which preserves the property (3). Thus the existence of left adjoints in (1) and (2) is equivalent to the existence of right adjoints.

Remark 6.8. The equivalent conditions in Proposition 6.6 are automatically satisfied if the algebra A is of finite representation type; see Theorem 2.5.

Proof of Proposition 6.6. (1) $\Leftrightarrow$ (2): See [40, §2].

(2) $\Rightarrow$ (3): A left adjoint $F \colon \operatorname{mod} A \to H^0\mathsf{U}$ sends a projective generator to a projective generator. Thus one takes $B = \operatorname{End}_A(FA)$ and gets an equivalence

$$\operatorname{Hom}_A(FA, -) \colon H^0\mathsf{U} \xrightarrow{\sim} \operatorname{mod} B.$$

Clearly, $\operatorname{mod} B$ has finite height and only finitely many simple objects.

(3) $\Rightarrow$ (4): It follows from [22, 8.2] that the category $H^0\mathsf{U}$ is equivalent to $\operatorname{mod} B$ for some finite dimensional k-algebra B. The algebra B is hereditary since A is hereditary. Thus the simple B-modules form a complete exceptional sequence $(X_1, \ldots, X_r)$ in $\operatorname{mod} B$. This gives an exceptional sequence in $\operatorname{mod} A$ satisfying $\mathsf{U} = \operatorname{Thick}(X_1, \ldots, X_r)$.

$(4) \Rightarrow (1)$: See [12, Theorem 3.2].

(1)–$(4) \Rightarrow (5)$: The duality provides a right adjoint for the inclusion of U. Thus the inclusion of $^{\perp}\mathsf{U}$ admits a left adjoint by Lemma 2.4. It follows that there is an exceptional sequence $(X_{r+1}, \ldots, X_s)$ in $\operatorname{mod} A$ with $^{\perp}\mathsf{U} = \operatorname{Thick}(X_{r+1}, \ldots, X_s)$. Then $(X_1, \ldots, X_s)$ is an exceptional sequence satisfying $\operatorname{Thick}(X_1, \ldots, X_s) = \mathbf{D}^b(\operatorname{mod} A)$ which is therefore complete.

$(5) \Rightarrow (4)$: Clear. $\qquad\square$

Example 6.9. For a tame hereditary algebra A, the regular A-modules form an exact abelian and extension closed subcategory of $\operatorname{mod} A$ that is of infinite height.

A classification. The following result is a consequence of Theorem 6.3 and provides a combinatorial classification of the thick subcategories of $\mathbf{D}^b(\operatorname{mod} A)$ satisfying the equivalent conditions in Proposition 6.6. Using the correspondence in Proposition 6.5, this translates into a classification of certain abelian subcategories of $\operatorname{mod} A$.

For the path algebra of a quiver over an algebraically closed field, this result is due to Igusa, Schiffler, and Thomas [28], Previous work of Ingalls and Thomas [29] establishes the result in finite and affine type.

Theorem 6.10. *Let A be a connected hereditary Artin algebra with simple modules $S_1, \ldots, S_n$ satisfying $\operatorname{Ext}^1_A(S_j, S_i) = 0$ for all $i < j$. Denote by W the associated Weyl group and fix the Coxeter element $c = s_1 \cdots s_n$. Then there exists an order preserving bijection between*

— *the set of thick subcategories of $\mathbf{D}^b(\operatorname{mod} A)$ such that the inclusion admits a left adjoint, and*

— *the set of non-crossing partitions $\operatorname{NC}(W, c)$.*

The map sends a thick subcategory which is generated by an exceptional sequence $(X_1, \ldots, X_r)$ to $s_{X_1} \cdots s_{X_r}$.

Let us formulate an immediate consequence.

Corollary 6.11. *Let $(X_1, \ldots, X_r)$ and $(Y_1, \ldots, Y_s)$ be exceptional sequences in $\operatorname{mod} A$. Then*

$$\operatorname{Thick}(X_1, \ldots, X_r) = \operatorname{Thick}(Y_1, \ldots, Y_s) \quad \Longleftrightarrow \quad s_{X_1} \cdots s_{X_r} = s_{Y_1} \cdots s_{Y_s}. \quad \square$$

Proof of Theorem 6.10. Fix a thick subcategory $\mathsf{U} \subseteq \mathbf{D}^b(\operatorname{mod} A)$ such that the inclusion admits a left adjoint. There exists a complete exceptional sequence $(X_1, \ldots, X_n)$ in $\operatorname{mod} A$ such that $\mathsf{U} = \operatorname{Thick}(X_1, \ldots, X_r)$ for some $r \leq n$, by Proposition 6.6. We assign to U the element $\operatorname{cox}(\mathsf{U}) = s_{X_1} \cdots s_{X_r}$ in W. Observe that $\operatorname{cox}(\mathsf{U}) \leq c$ by Lemma 6.1, since $s_{X_1} \cdots s_{X_n} = c$ by Theorem 6.3. Thus cox gives a map into $\operatorname{NC}(W, c)$.

The map cox *is well-defined:* Choose a second exceptional sequence $(Y_1, \ldots, Y_s)$ in $\operatorname{mod} A$ such that $\mathsf{U} = \operatorname{Thick}(Y_1, \ldots, Y_s)$. Then $(Y_1, \ldots, Y_s, X_{r+1}, \ldots, X_n)$ is a complete exceptional sequence, and it follows from Theorem 6.3 that

$$s_{Y_1} \cdots s_{Y_s} s_{X_{r+1}} \cdots s_{X_n} = c = s_{X_1} \cdots s_{X_r} s_{X_{r+1}} \cdots s_{X_n}.$$

Thus $s_{Y_1} \cdots s_{Y_s} = s_{X_1} \cdots s_{X_r}$.

The map cox *is injective:* Let U and V be thick subcategories such that $\operatorname{cox}(\mathsf{U}) = \operatorname{cox}(\mathsf{V})$. Thus there are two complete exceptional sequences $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$ such that $\mathsf{U} = \operatorname{Thick}(X_1, \ldots, X_r)$ and $\mathsf{V} = \operatorname{Thick}(Y_1, \ldots, Y_s)$ for some pair of integers $r, s \leq n$. Moreover, $s_{X_1} \cdots s_{X_r} = s_{Y_1} \cdots s_{Y_s}$. It follows that

$$r = \ell(s_{X_1} \cdots s_{X_r}) = \ell(s_{Y_1} \cdots s_{Y_s}) = s$$

and therefore $c = s_{Y_1} \cdots s_{Y_s} s_{X_{r+1}} \cdots s_{X_n}$. From Corollary 6.4 and Lemma 6.2 one gets that $(Y_1, \ldots, Y_s, X_{r+1}, \ldots, X_n)$ is a complete exceptional sequence. Thus

$$\mathsf{U} = \mathrm{Thick}(X_1, \ldots, X_r) = \mathrm{Thick}(X_{r+1}, \ldots, X_n)^{\perp} = \mathrm{Thick}(Y_1, \ldots, Y_s) = \mathsf{V}.$$

The map cox *is surjective:* Let $w \in NC(W, c)$. Thus we can write

$$c = x_1 \cdots x_r x_{r+1} \cdots x_n$$

as products of reflections $x_i \in W_1$ such that $w = x_1 \cdots x_r$. From Corollary 6.4 one gets a complete exceptional sequence $(X_1, \ldots, X_n)$ with $x_i = s_{X_i}$ for all i. Set $\mathsf{U} = \mathrm{Thick}(X_1, \ldots, X_r)$. Then $\mathrm{cox}(\mathsf{U}) = w$.

The map cox *is order preserving:* Let $\mathsf{V} \subseteq \mathsf{U}$ be thick subcategories. From Proposition 6.6 one gets a complete exceptional sequence $(X_1, \ldots, X_n)$ such that

$$\mathsf{V} = \mathrm{Thick}(X_1, \ldots, X_s) \quad \text{and} \quad \mathsf{U} = \mathrm{Thick}(X_1, \ldots, X_r)$$

for some $s \le r \le n$. It follows from Lemma 6.1 that

$$\mathrm{cox}(\mathsf{V}) = s_{X_1} \cdots s_{X_s} \le s_{X_1} \cdots s_{X_r} = \mathrm{cox}(\mathsf{U}). \qquad \square$$

Remark 6.12. Let $\mathsf{U} \subseteq \mathbf{D}^b(\mathrm{mod}\, A)$ be a thick subcategory such that the inclusion admits a left adjoint. Then $\mathrm{cox}(\mathsf{U}) \, \mathrm{cox}(^{\perp}\mathsf{U}) = c$.

Example: The Kronecker algebra. Let k be a field and consider the *Kronecker algebra*, that is, the path algebra of the quiver $\circ \rightrightarrows \circ$. This is a tame hereditary Artin algebra; we denote it by K and compute the lattice of thick subcategories of $\mathbf{D}^b(\mathrm{mod}\, K)$. For a description of $\mathrm{mod}\, K$, we refer to [6, VIII.7].

Each finite dimensional indecomposable K-module is either exceptional or regular. The dimension vectors of the exceptional K-modules are $(p, q) \in \mathbb{Z}^2$ with $p, q \ge 0$ and $|p - q| = 1$. Thus the non-crossing partitions with respect to the Coxeter element $c = s_1 s_2$ form the lattice

$$NC(W, c) = \{ s_{(p,q)} \mid p, q \ge 0 \text{ and } |p - q| = 1 \} \cup \{ \mathrm{id}, c \}$$

with the following Hasse diagram:
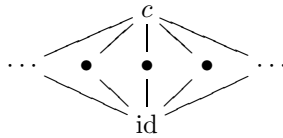
The regular K-modules form an extension closed exact abelian subcategory of $\mathrm{mod}\, K$ that is uniserial. The simple objects of this abelian category are parameterised by the closed points of the projective line over k, which we identify with non-maximal and non-zero homogeneous prime ideals $\mathfrak{p} \subseteq k[x, y]$. We denote the set of closed points by $\mathbb{P}^1(k)$ and write $\mathbf{2}^{\mathbb{P}^1(k)}$ for its power set. Adding an extra greatest element (the set of all non-maximal homogeneous prime ideals) to $\mathbf{2}^{\mathbb{P}^1(k)}$ yields a new lattice which we denote by $\widehat{\mathbf{2}}^{\mathbb{P}^1(k)}$. The simple object corresponding to $\mathfrak{p}$ is denoted by $S_{\mathfrak{p}}$. We obtain an injective map

$$\widehat{\mathbf{2}}^{\mathbb{P}^1(k)} \longrightarrow \mathbf{T}(\mathbf{D}^b(\mathrm{mod}\, K))$$

by sending $U \subseteq \mathbb{P}^1(k)$ to $\mathrm{Thick}(\{ S_{\mathfrak{p}} \mid \mathfrak{p} \in U \})$ and $\mathbb{P}^1(k) \cup \{0\}$ to $\mathbf{D}^b(\mathrm{mod}\, K)$.

Let L', L'' be a pair of lattices with smallest elements $0', 0''$ and greatest elements $1', 1''$. Denote by $L' \amalg L''$ the new lattice which is obtained from the disjoint union $L' \cup L''$ (viewed as sum of posets) by identifying $0' = 0''$ and $1' = 1''$.

Proposition 6.13. *The lattice of thick subcategories of* $\mathbf{D}^b(\operatorname{mod} K)$ *is isomorphic to the lattice* $\operatorname{NC}(W, c) \amalg \widehat{\mathbf{2}}^{\mathbb{P}^1(k)}$.

Proof. We write $\mathbf{T}(K) = \mathbf{T}(\mathbf{D}^b(\operatorname{mod} K))$ and have injective maps

$$\operatorname{NC}(W, c) \longrightarrow \mathbf{T}(K) \quad \text{and} \quad \widehat{\mathbf{2}}^{\mathbb{P}^1(k)} \longrightarrow \mathbf{T}(K)$$

which induce an order preserving map

$$\operatorname{NC}(W, c) \amalg \widehat{\mathbf{2}}^{\mathbb{P}^1(k)} \longrightarrow \mathbf{T}(K).$$

In order to prove bijectivity, fix a pair of indecomposable K-modules X, Y such that X is exceptional and Y is regular. We have $\operatorname{Thick}(X) \cap \operatorname{Thick}(Y) = 0$ and this gives injectivity; surjectivity follows from the fact that $\operatorname{Thick}(X, Y) = \mathbf{D}^b(\operatorname{mod} K)$. $\quad\square$

The category of coherent sheaves on the projective line $\mathbb{P}^1_k$ admits a tilting object $T = \mathcal{O}(0) \oplus \mathcal{O}(1)$ such that $\operatorname{End}(T) \cong K$; see [8]. Thus $\mathbf{R}\operatorname{Hom}(T, -)$ induces a triangle equivalence

$$\mathbf{D}^b(\operatorname{coh} \mathbb{P}^1_k) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod} K)$$

and this yields a description of the lattice of thick subcategories of $\mathbf{D}^b(\operatorname{coh} \mathbb{P}^1_k)$. Note that the category of coherent sheaves carries a tensor product. The thick tensor subcategories have been classified by Thomason [46, Theorem 3.5]; they are precisely the ones parameterised by subsets of $\mathbb{P}^1(k)$.

Appendix A. Auslander–Reiten theory

In this appendix we collect some basic facts from Auslander–Reiten theory, as initiated by Auslander and Reiten in [5]. Krull–Remak–Schmidt categories form the appropriate setting for this theory, while exact or triangulated structures are irrelevant for most parts; see also [7, 41]. This material is well-known, at least for categories that are linear over a field with finite dimensional morphism spaces. We provide full proofs for most statements, including references whenever they are available.

Let C be an essentially small additive category. Then C is called *Krull–Remak–Schmidt category* if every object in C is a finite coproduct of indecomposable objects with local endomorphism rings.

It is convenient to view C as a ring with several objects. Thus we use the category $\operatorname{Mod} \mathsf{C}$ of C-*modules*, which are by definition the additive functors $\mathsf{C}^{\operatorname{op}} \to \mathsf{Ab}$ into the category of abelian groups.

There is the following useful characterisation in terms of projective covers. Recall that a morphism $\phi\colon P \to M$ is a *projective cover*, if P is a projective object and ϕ is an *essential epimorphism*, that is, a morphism $\alpha\colon P' \to P$ is an epimorphism if and only if $\phi\alpha$ is an epimorphism.

Proposition A.1. *For an essentially small additive category* C *with split idempotents the following are equivalent.*

(1) *The category* C *is a Krull–Remak–Schmidt category.*
(2) *Every finitely generated* C*-module admits a projective cover.*

The proof requires some preparations, and we begin with two lemmas.

Lemma A.2. *Let* P *be a projective object. Then the following are equivalent for an epimorphism* $\phi\colon P \to M$.

(1) *The morphism* ϕ *is a projective cover of* M.
(2) *Every endomorphism* $\alpha\colon P \to P$ *satisfying* $\phi\alpha = \phi$ *is an isomorphism.*

Proof. (1) $\Rightarrow$ (2): Let $\alpha \colon P \to P$ be an endomorphism satisfying $\phi\alpha = \phi$. Then α is an epimorphism since ϕ is essential. Thus there exists $\alpha' \colon P \to P$ satisfying $\alpha\alpha' = \mathrm{id}_P$ since P is projective. It follows that $\phi\alpha' = \phi$ and therefore α' is an epimorphism. On the other hand, α' is a monomorphism. Thus α' and α are isomorphisms.

(2) $\Rightarrow$ (1): Let $\alpha \colon P' \to P$ be a morphism such that $\phi\alpha$ is an epimorphism. Then ϕ factors through $\phi\alpha$ via a morphism $\alpha' \colon P \to P'$ since P is projective. The composite $\alpha\alpha'$ is an isomorphism and therefore α is an epimorphism. Thus ϕ is essential. $\qquad\square$

Lemma A.3. *Let $\phi \colon P \to S$ be an epimorphism such that P is projective and S is simple. Then the following are equivalent.*

(1) *The morphism ϕ is a projective cover of S.*
(2) *The object P has a unique maximal subobject.*
(3) *The endomorphism ring of P is local.*

Proof. (1) $\Rightarrow$ (2): Let $U \subseteq P$ be a subobject and suppose $U \not\subseteq \mathrm{Ker}\,\phi$. Then $U + \mathrm{Ker}\,\phi = P$, and therefore $U = P$ since ϕ is essential. Thus $\mathrm{Ker}\,\phi$ contains every proper subobject of P.

(2) $\Rightarrow$ (3): First observe that P is an indecomposable object. It follows that every endomorphism of P is invertible if and only if it is an epimorphism. Given two non-units α, β in $\mathrm{End}(P)$, we have therefore $\mathrm{Im}(\alpha + \beta) \subseteq \mathrm{Im}\,\alpha + \mathrm{Im}\,\beta \subsetneq P$. Here we use that P has a unique maximal subobject. Thus $\alpha + \beta$ is a non-unit and $\mathrm{End}(P)$ is local.

(3) $\Rightarrow$ (1): Consider the $\mathrm{End}(P)$-submodule H of $\mathrm{Hom}(P, S)$ which is generated by ϕ. Suppose $\phi = \phi\alpha$ for some α in $\mathrm{End}(P)$. If α belongs to the Jacobson radical, then $H = HJ(\mathrm{End}(P))$, which is not possible by Nakayama's lemma. Thus α is an isomorphism since $\mathrm{End}(P)$ is local. It follows from Lemma A.2 that ϕ is a projective cover. $\qquad\square$

Given any C-module M, we denote by $\mathrm{rad}\,M$ its *radical*, that is, the intersection of all maximal submodules of M.

Proof of Proposition A.1. First observe that the Yoneda functor $\mathsf{C} \to \mathrm{Mod}\,\mathsf{C}$ taking an object X to $\mathrm{Hom}_{\mathsf{C}}(-, X)$ identifies C with the category of finitely generated projective C-modules. The other tools are Lemmas A.2 and A.3, which are used without further reference.

(1) $\Rightarrow$ (2): Let M be a simple C-module and choose an indecomposable object X with $M(X) \neq 0$. Then there exists a non-zero morphism $\mathrm{Hom}_{\mathsf{C}}(-, X) \to M$ which is a projective cover since $\mathrm{End}_{\mathsf{C}}(X)$ is local. Thus every finite sum of simple C-modules admits a projective cover.

Now let M be a finitely generated C-module and choose an epimorphism $\phi \colon P \to M$ with P finitely generated projective. Let $P = \bigoplus_i P_i$ be a decomposition into indecomposable modules. Then

$$P/\mathrm{rad}\,P = \bigoplus_i P_i/\mathrm{rad}\,P_i$$

is a finite sum of simple C-modules since each P_i has a local endomorphism ring. The epimorphism ϕ induces an epimorphism $P/\mathrm{rad}\,P \to M/\mathrm{rad}\,M$ and therefore $M/\mathrm{rad}\,M$ decomposes into finitely many simple modules. There exists a projective cover $Q \to M/\mathrm{rad}\,M$ and this factors through the canonical morphism $\pi \colon M \to M/\mathrm{rad}\,M$ via a morphism $\psi \colon Q \to M$. The morphism π is essential since M is

finitely generated. It follows that ψ is an epimorphism. The morphism ψ is essential since $\pi\psi$ is essential. Thus ψ is a projective cover.

(2) $\Rightarrow$ (1): Let X be an object in C. We show that X admits a decomposition into finitely many indecomposable objects. Set $P = \mathrm{Hom}_\mathsf{C}(-, X)$. We claim that $P/\operatorname{rad}P$ is semisimple. Choose a quotient P/U, where $\operatorname{rad}P \subseteq U \subseteq P$, and let $Q \to P/U$ be a projective cover. This gives maps $P \to Q$ and $Q \to P$. The composite $Q \to P \to Q$ is invertible and induces an isomorphism $P/U \to P/\operatorname{rad}P \to P/U$. Thus the canonical morphism $P/\operatorname{rad}P \to P/U$ has a right inverse, and we conclude that $P/\operatorname{rad}P$ is semisimple. Let $P/\operatorname{rad}P = \bigoplus_i S_i$ be a decomposition into finitely many simple objects and choose a projective cover $P_i \to S_i$ for each i. Then $P \cong \bigoplus_i P_i$, since $P \to P/\operatorname{rad}P$ and $\bigoplus_i P_i \to \bigoplus_i S_i$ are both projective covers. It remains to observe that each P_i is indecomposable with a local endomorphism ring. $\qquad\square$

Remark A.4. Fix a ring A and let C be the category of finitely generated projective A-modules. Then the functor $\operatorname{Mod}\mathsf{C} \to \operatorname{Mod}A$ taking X to $X(A)$ is an equivalence. The category C is Krull–Remak–Schmidt if and only if the ring A is semiperfect.

From now on suppose that C is a Krull–Remak–Schmidt category.

The radical. The *radical* of C is by definition the collection of subgroups

$$\operatorname{Rad}_\mathsf{C}(X, Y) \subseteq \mathrm{Hom}_\mathsf{C}(X, Y)$$

for each pair X, Y of objects in C, where

$$\operatorname{Rad}_\mathsf{C}(X, Y) = \{\phi \in \mathrm{Hom}_\mathsf{C}(X, Y) \mid \mathrm{id}_X - \phi'\phi \text{ is invertible for all } \phi'\colon Y \to X\}$$
$$= \{\phi \in \mathrm{Hom}_\mathsf{C}(X, Y) \mid \mathrm{id}_Y - \phi\phi' \text{ is invertible for all } \phi'\colon Y \to X\}.$$

The radical is a *two-sided ideal* of C, that is, a subfunctor of $\mathrm{Hom}_\mathsf{C}(-, -)\colon \mathsf{C}^\mathrm{op} \times \mathsf{C} \to \mathsf{Ab}$. It is actually the unique two-sided ideal $\mathfrak{J}$ of C such that $\mathfrak{J}(X, X)$ equals the Jacobson radical of $\mathrm{End}_\mathsf{C}(X)$ for each object X; see [34].

Given two decompositions $X = \coprod_i X_i$ and $Y = \coprod_j Y_j$ in C, we have

$$\operatorname{Rad}_\mathsf{C}(X, Y) = \bigoplus_{i,j} \operatorname{Rad}_\mathsf{C}(X_i, Y_j).$$

A morphism between indecomposable objects belongs to the radical if and only if it is not invertible, since indecomposable objects have local endomorphism rings.

Simple functors. Given an object X in C, the functor $\operatorname{Rad}_\mathsf{C}(-, X)$ equals the intersection of all maximal subfunctors of $\mathrm{Hom}_\mathsf{C}(-, X)\colon \mathsf{C}^\mathrm{op} \to \mathsf{Ab}$. This observation has the following consequence, where

$$S_X = \mathrm{Hom}_\mathsf{C}(-, X)/\operatorname{Rad}_\mathsf{C}(-, X).$$

Lemma A.5 ([3, Proposition 2.3]). *The map sending an object X of C to the functor S_X induces (up to isomorphism) a bijection between the indecomposable objects of C and the simple objects of $\operatorname{Mod}\mathsf{C}$.*

Proof. If an object X has a local endomorphism ring, then $\operatorname{Rad}_\mathsf{C}(-, X)$ is the unique maximal subobject of $\mathrm{Hom}_\mathsf{C}(-, X)$. Thus S_X is simple in that case, and the inverse map sends a simple object S in $\operatorname{Mod}\mathsf{C}$ to the unique indecomposable object X in C such that $S(X) \neq 0$. $\qquad\square$

Irreducible morphisms. The definition of the radical is extended recursively as follows. For each $n > 1$ and each pair of objects X, Y let $\mathrm{Rad}_{\mathsf{C}}^n(X, Y)$ be the set of morphisms $\phi \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ that admit a factorisation $\phi = \phi'' \phi'$ with $\phi' \in \mathrm{Rad}_{\mathsf{C}}(X, Z)$ and $\phi'' \in \mathrm{Rad}_{\mathsf{C}}^{n-1}(Z, Y)$ for some object Z. Then we set

$$\mathrm{Irr}_{\mathsf{C}}(X, Y) = \mathrm{Rad}_{\mathsf{C}}(X, Y) / \mathrm{Rad}_{\mathsf{C}}^2(X, Y).$$

This is a bimodule over the rings $\Delta(X)$ and $\Delta(Y)$, where

$$\Delta(X) = \mathrm{End}_{\mathsf{C}}(X) / \mathrm{Rad}_{\mathsf{C}}(X, X).$$

Note that a morphism $X \to Y$ between indecomposable objects belongs to $\mathrm{Rad}_{\mathsf{C}}(X, Y) \smallsetminus \mathrm{Rad}_{\mathsf{C}}^2(X, Y)$ if and only it is irreducible. A morphism ϕ is called *irreducible* if ϕ is neither a section nor a retraction, and if $\phi = \phi'' \phi'$ is a factorisation then ϕ' is a section or ϕ'' is a retraction.

Lemma A.6. *For each pair of indecomposable objects X, Y in C, we have*

$$\mathrm{Ext}_{\mathsf{C}}^1(S_Y, S_X) \cong \mathrm{Hom}_{\Delta(X)}(\mathrm{Irr}_{\mathsf{C}}(X, Y), \Delta(X))$$

as bimodules over $\Delta(X)$ and $\Delta(Y)$.

Proof. Applying $\mathrm{Hom}_{\mathsf{C}}(-, S_X)$ to the exact sequence

$$0 \longrightarrow \mathrm{Rad}_{\mathsf{C}}(-, Y) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(-, Y) \longrightarrow S_Y \longrightarrow 0$$

gives

$$\mathrm{Ext}_{\mathsf{C}}^1(S_Y, S_X) \cong \mathrm{Hom}_{\mathsf{C}}(\mathrm{Rad}_{\mathsf{C}}(-, Y), S_X).$$

Then applying $\mathrm{Hom}_{\mathsf{C}}(-, S_X)$ to the exact sequence

$$0 \longrightarrow \mathrm{Rad}_{\mathsf{C}}^2(-, Y) \longrightarrow \mathrm{Rad}_{\mathsf{C}}(-, Y) \longrightarrow \mathrm{Irr}_{\mathsf{C}}(-, Y) \longrightarrow 0$$

gives

$$\mathrm{Ext}_{\mathsf{C}}^1(S_Y, S_X) \cong \mathrm{Hom}_{\mathsf{C}}(\mathrm{Irr}_{\mathsf{C}}(-, Y), S_X) \cong \mathrm{Hom}_{\Delta(X)}(\mathrm{Irr}_{\mathsf{C}}(X, Y), \Delta(X))$$

since $S_X(X) = \Delta(X)$. $\qquad\square$

Almost split morphisms. A morphism $\phi \colon X \to Y$ is called *right almost split* if ϕ is not a retraction and if every morphism $X' \to Y$ that is not a retraction factors through ϕ. The morphism ϕ is *right minimal* if every endomorphism $\alpha \colon X \to X$ with $\phi\alpha = \phi$ is invertible. Note that Y is indecomposable if ϕ is right almost split. *Left almost split* morphisms and *left minimal* morphisms are defined dually. The term *minimal right almost split* means *right minimal* and *right almost split*.

Recall that a projective presentation

$$P_n \xrightarrow{\delta_n} P_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} M \longrightarrow 0$$

is *minimal* if each morphism $P_i \to \mathrm{Im}\,\delta_i$ is a projective cover.

Lemma A.7 ([4, Chap. II, Proposition 2.7]). *A morphism $X \to Y$ in C is minimal right almost split if and only if it induces in $\mathrm{Mod}\,\mathsf{C}$ a minimal projective presentation*

$$\mathrm{Hom}_{\mathsf{C}}(-, X) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(-, Y) \longrightarrow S \longrightarrow 0$$

of a simple object.

Proof. A non-zero morphism $\mathrm{Hom}_{\mathsf{C}}(-, Y) \to S$ to a simple object is a projective cover if and only if $\mathrm{End}_{\mathsf{C}}(Y)$ is local. In that case the exactness of the sequence means that the image of $\mathrm{Hom}_{\mathsf{C}}(-, X) \to \mathrm{Hom}_{\mathsf{C}}(-, Y)$ equals $\mathrm{Rad}_{\mathsf{C}}(-, Y)$; it is therefore equivalent to the fact that $X \to Y$ is right almost split. The canonical morphism from $\mathrm{Hom}_{\mathsf{C}}(-, X)$ to the image of $\mathrm{Hom}_{\mathsf{C}}(-, X) \to \mathrm{Hom}_{\mathsf{C}}(-, Y)$ is a projective cover if and only if $X \to Y$ is right minimal. $\qquad\square$

Almost split morphisms and irreducible morphisms are related as follows.

Lemma A.8 ([7, Corollary 3.4]). *Let $X \to Y$ be a minimal right almost split morphism in C and let $X = X_1^{n_1} \amalg \ldots \amalg X_r^{n_r}$ be a decomposition into indecomposable objects such that the X_i are pairwise non-isomorphic. Given an indecomposable object X', one has $\mathrm{Irr}_{\mathsf{C}}(X', Y) \neq 0$ if and only if $X' \cong X_i$ for some i. In that case n_i equals the length of $\mathrm{Irr}_{\mathsf{C}}(X', Y)$ over $\Delta(X')$.*

Proof. The morphism $X \to Y$ induces a minimal projective presentation

$$\mathrm{Hom}_{\mathsf{C}}(-, X) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(-, Y) \longrightarrow S_Y \longrightarrow 0$$

by Lemma A.7, and therefore a projective cover

$$\pi \colon \mathrm{Hom}_{\mathsf{C}}(-, X) \longrightarrow \mathrm{Rad}_{\mathsf{C}}(-, Y).$$

This morphism induces an isomorphism

$$\mathrm{Hom}_{\mathsf{C}}(-, X)/\mathrm{Rad}_{\mathsf{C}}(-, X) \xrightarrow{\sim} \mathrm{Rad}_{\mathsf{C}}(-, Y)/\mathrm{Rad}_{\mathsf{C}}^2(-, Y)$$

since $\mathrm{Ker}\,\pi \subseteq \mathrm{Rad}_{\mathsf{C}}(-, X)$. On the other hand, the decomposition of X implies

$$\mathrm{Hom}_{\mathsf{C}}(-, X)/\mathrm{Rad}_{\mathsf{C}}(-, X) \cong S_{X_1}^{n_1} \amalg \ldots \amalg S_{X_r}^{n_r}.$$

It remains to observe that $S_{X_i}(X') \neq 0$ if and only if $X' \cong X_i$. $\square$

Almost split sequences. The following definition of an almost split sequence is taken from Liu [41]; it covers the original definition of Auslander and Reiten for abelian categories [5], but also Happel's definition of an Auslander–Reiten triangle in a triangulated category [24].

A sequence of morphisms $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ in C is called *almost split* if

(1) α is minimal left almost split and a weak kernel of β,
(2) β is minimal right almost split and a weak cokernel of α, and
(3) $Y \neq 0$.

The end terms X and Z determine each other up to isomophism, and we write $X = \tau Z$ and $Z = \tau^{-1} X$. One calls τZ the *Auslander–Reiten translate* of Z.

Lemma A.9. *A sequence of morphisms $X \to Y \to Z$ in C is almost split if and only if it induces two minimal projective presentations*

$$\mathrm{Hom}_{\mathsf{C}}(-, X) \to \mathrm{Hom}_{\mathsf{C}}(-, Y) \to \mathrm{Hom}_{\mathsf{C}}(-, Z) \to S_Z \to 0$$
$$\mathrm{Hom}_{\mathsf{C}}(Z, -) \to \mathrm{Hom}_{\mathsf{C}}(Y, -) \to \mathrm{Hom}_{\mathsf{C}}(X, -) \to S^X \to 0$$

where we use the notation

$$S^X = \mathrm{Hom}_{\mathsf{C}}(X, -)/\mathrm{Rad}_{\mathsf{C}}(X, -).$$

Proof. Apply Lemma A.7. $\square$

The Auslander–Reiten translate is functorial in the following sense.

Lemma A.10. *Let X, Y be indecomposable objects in C and suppose their Auslander–Reiten translates are defined. Then $\Delta(X) \cong \Delta(\tau X)$ and $\Delta(Y) \cong \Delta(\tau Y)$ as division rings. Using these isomorphisms, we have*

$$\mathrm{Hom}_{\Delta(X)}(\mathrm{Irr}_{\mathsf{C}}(X, Y), \Delta(X)) \cong \mathrm{Hom}_{\Delta(\tau Y)}(\mathrm{Irr}_{\mathsf{C}}(\tau X, \tau Y), \Delta(\tau Y))$$

as bimodules over $\Delta(X)$ and $\Delta(Y)$.

Proof. We use the *stable category* of finitely presented C-modules which we denote by $\underline{\mathrm{mod}}\,\mathsf{C}$. The objects are the C-modules M that admit a presentation

$$(*) \qquad\qquad \mathrm{Hom}_{\mathsf{C}}(-, X) \xrightarrow{(-,\phi)} \mathrm{Hom}_{\mathsf{C}}(-, Y) \longrightarrow M \longrightarrow 0$$

and the morphisms are all C-linear morphisms modulo the subgroup of morphisms that factor through a projective C-module. There are two functors

$$\Omega\colon \underline{\mathrm{mod}}\,\mathsf{C} \longrightarrow \underline{\mathrm{mod}}\,\mathsf{C} \quad \text{and} \quad \mathrm{Tr}\colon \underline{\mathrm{mod}}\,\mathsf{C} \longrightarrow \underline{\mathrm{mod}}\,\mathsf{C}^{\mathrm{op}}$$

taking a module to its *syzygy* and its *transpose*, respectively. Both functors are defined for a module M with presentation $(*)$ via exact sequences as follows:

$$0 \longrightarrow \Omega M \longrightarrow \mathrm{Hom}_{\mathsf{C}}(-, Y) \longrightarrow M \longrightarrow 0$$

$$\mathrm{Hom}_{\mathsf{C}}(Y, -) \xrightarrow{(\phi, -)} \mathrm{Hom}_{\mathsf{C}}(X, -) \longrightarrow \mathrm{Tr}\, M \longrightarrow 0$$

Note that the transpose yields an equivalence $(\underline{\mathrm{mod}}\,\mathsf{C})^{\mathrm{op}} \xrightarrow{\sim} \underline{\mathrm{mod}}\,\mathsf{C}^{\mathrm{op}}$.

Using the minimal projective presentations of S_X and $S^{\tau X}$ from Lemma A.9, one gets isomorphisms

$$\Omega S_X \cong \mathrm{Tr}\, S^{\tau X} \qquad \text{and} \qquad \Omega S^{\tau X} \cong \mathrm{Tr}\, S_X.$$

These yield mutually inverse maps

$$\Delta(X) \xrightarrow{\sim} \underline{\mathrm{End}}_{\mathsf{C}}(S_X) \to \underline{\mathrm{End}}_{\mathsf{C}}(\Omega S_X) \xrightarrow{\sim}$$
$$\underline{\mathrm{End}}_{\mathsf{C}}(\mathrm{Tr}\, S^{\tau X}) \xrightarrow{\sim} \underline{\mathrm{End}}_{\mathsf{C}^{\mathrm{op}}}(S^{\tau X})^{\mathrm{op}} \xrightarrow{\sim} \Delta(\tau X)$$

and

$$\Delta(\tau X) \xrightarrow{\sim} \underline{\mathrm{End}}_{\mathsf{C}^{\mathrm{op}}}(S^{\tau X})^{\mathrm{op}} \to \underline{\mathrm{End}}_{\mathsf{C}^{\mathrm{op}}}(\Omega S^{\tau X})^{\mathrm{op}} \xrightarrow{\sim}$$
$$\underline{\mathrm{End}}_{\mathsf{C}^{\mathrm{op}}}(\mathrm{Tr}\, S_X)^{\mathrm{op}} \xrightarrow{\sim} \underline{\mathrm{End}}_{\mathsf{C}}(S_X) \xrightarrow{\sim} \Delta(X).$$

Next we apply Lemma A.6 and obtain

$$\mathrm{Hom}_{\Delta(X)}(\mathrm{Irr}_{\mathsf{C}}(X, Y), \Delta(X)) \cong \mathrm{Ext}^1_{\mathsf{C}}(S_Y, S_X)$$
$$\cong \underline{\mathrm{Hom}}_{\mathsf{C}}(\Omega S_Y, S_X)$$
$$\cong \underline{\mathrm{Hom}}_{\mathsf{C}}(\mathrm{Tr}\, S^{\tau Y}, S_X)$$
$$\cong \underline{\mathrm{Hom}}_{\mathsf{C}^{\mathrm{op}}}(\mathrm{Tr}\, S_X, S^{\tau Y})$$
$$\cong \underline{\mathrm{Hom}}_{\mathsf{C}^{\mathrm{op}}}(\Omega S^{\tau X}, S^{\tau Y})$$
$$\cong \mathrm{Ext}^1_{\mathsf{C}^{\mathrm{op}}}(S^{\tau X}, S^{\tau Y})$$
$$\cong \mathrm{Hom}_{\Delta(\tau X)}(\mathrm{Irr}_{\mathsf{C}^{\mathrm{op}}}(\tau Y, \tau X), \Delta(\tau X))$$
$$\cong \mathrm{Hom}_{\Delta(\tau X)}(\mathrm{Irr}_{\mathsf{C}}(\tau X, \tau Y), \Delta(\tau X)).$$

The second isomorphism requires an extra argument, and the same is used for the sixth. Let $\Omega S_Y = \mathrm{Rad}_{\mathsf{C}}(-, Y)$, and observe that this has no non-zero projective direct summand by the minimality of the presentations in Lemma A.9. Then we have

$$\mathrm{Ext}^1_{\mathsf{C}}(S_Y, S_X) \cong \mathrm{Hom}_{\mathsf{C}}(\Omega S_Y, S_X) \cong \underline{\mathrm{Hom}}_{\mathsf{C}}(\Omega S_Y, S_X).$$

The first isomorphism is from the proof of Lemma A.6. The second follows from the fact that any non-zero morphism $\Omega S_Y \to S_X$ factoring through a projective factors through the projective cover $\mathrm{Hom}_{\mathsf{C}}(-, X) \to S_X$. This means that $\mathrm{Hom}_{\mathsf{C}}(-, X)$ is a direct summand of ΩS_Y, which has been excluded before. $\square$

The Auslander–Reiten quiver. The *Auslander–Reiten quiver* of C is defined as follows. The set of isomorphism classes of indecomposable objects in C form the *vertices*, and there is an *arrow* $X \to Y$ if $\mathrm{Irr}_\mathsf{C}(X, Y) \neq 0$. It is often convenient to identify an indecomposable object with its isomorphism class.

The Auslander–Reiten quiver carries a *valuation* which assigns to each arrow $X \to Y$ the pair $(\delta_{X,Y}, \delta'_{X,Y})$, where

$$\delta_{X,Y} = \ell_{\Delta(X)}(\mathrm{Irr}_\mathsf{C}(X,Y)) \quad \text{and} \quad \delta'_{X,Y} = \ell_{\Delta(Y)}(\mathrm{Irr}_\mathsf{C}(X,Y)).$$

Here, $\ell_A(M)$ denotes the length of an A-module M.

Lemma A.11. *Let X, Y be indecomposable objects in C and suppose that τY is defined. Then $\delta'_{\tau Y, X} = \delta_{X,Y}$.*

Proof. We have an almost split sequence $\tau Y \to \bar{X} \to Y$ and $\delta_{X,Y}$ counts the multiplicity of X in a decomposition of $\bar{X}$ by Lemma A.8, which equals $\delta'_{\tau Y, X}$ by the dual of Lemma A.8. $\qquad\square$

Lemma A.12. *Let k be a commutative ring and suppose that C is k-linear with all morphism spaces of finite length over k. Let X, Y be indecomposable objects in C and suppose that τY is defined. Then $\delta_{\tau Y, X} = \delta'_{X,Y}$.*

Proof. Using the identity $\delta'_{\tau Y, X} = \delta_{X,Y}$ from Lemma A.11 and the isomorphism $\Delta(Y) \cong \Delta(\tau Y)$ from Lemma A.10, one computes

$$
\begin{aligned}
\ell_{\Delta(Y)}(\mathrm{Irr}_\mathsf{C}(X,Y)) &= \ell_k(\mathrm{Irr}_\mathsf{C}(X,Y)) \cdot \ell_k(\Delta(Y))^{-1} \\
&= \ell_{\Delta(X)}(\mathrm{Irr}_\mathsf{C}(X,Y)) \cdot \ell_k(\Delta(X)) \cdot \ell_k(\Delta(Y))^{-1} \\
&= \ell_{\Delta(X)}(\mathrm{Irr}_\mathsf{C}(\tau Y, X)) \cdot \ell_k(\Delta(X)) \cdot \ell_k(\Delta(Y))^{-1} \\
&= \ell_k(\mathrm{Irr}_\mathsf{C}(\tau Y, X)) \cdot \ell_k(\Delta(\tau Y))^{-1} \\
&= \ell_{\Delta(\tau Y)}(\mathrm{Irr}_\mathsf{C}(\tau Y, X)). \qquad\qquad\square
\end{aligned}
$$

The repetition. Let Γ be a quiver without loops. Then a new quiver $\mathbb{Z}\Gamma$ is defined as follows. The set of vertices is $\mathbb{Z} \times \Gamma_0$. For each arrow $x \to y$ in Γ and each $n \in \mathbb{Z}$ there is an arrow $(n, x) \to (n, y)$ and an arrow $(n-1, y) \to (n, x)$ in $\mathbb{Z}\Gamma$. The quiver $\mathbb{Z}\Gamma$ is a translation quiver [43] with translation τ defined by $\tau(n, x) = (n-1, x)$ for each vertex (n, x).

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